Math 2010   Week 5

Finding limit using polar coordinates

$(x, y) \longleftrightarrow (r, \theta)$ with $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

Note

$$\boxed{(x, y) \longrightarrow (0, 0) \iff r \to 0}$$

eg Find limits using polar coordinates

① $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3 + y^3}{x^2 + y^2}$

$= \lim\limits_{r\to 0} \dfrac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$

$= \lim\limits_{r\to 0} r(\cos^3\theta + \sin^3\theta)$

$= 0$ (Squeeze theorem)

Details

$$|\cos^3\theta + \sin^3\theta| \le |\cos^3\theta| + |\sin^3\theta| \le 2$$

$$\therefore \left| r(\cos^3\theta + \sin^3\theta) \right| \le 2|r|$$

Also $\lim\limits_{r\to 0} 2|r| = 0 \Rightarrow \lim\limits_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$

② $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 + xy}{2(x^2 + y^2)}$

$= \lim\limits_{r\to 0} \dfrac{r^2\cos^2\theta + r^2\cos\theta\sin\theta}{2r^2}$

$= \lim\limits_{r\to 0} \dfrac{\cos^2\theta + \cos\theta\sin\theta}{2}$

which depends on θ

eg $= \begin{cases} \dfrac{1}{2} & \text{if } \theta = 0 \\ 0 & \text{if } \theta = \dfrac{\pi}{2} \end{cases}$

$\therefore$ limit DNE
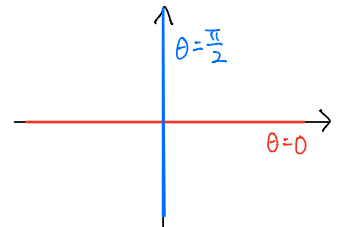
③ $\lim\limits_{(x,y)\to(0,0)} xy\ln(x^2+y^2)$

$= \lim\limits_{r\to 0} \underbrace{r^2\cos\theta\sin\theta \ln(r^2)}$

Note $|\cos\theta\sin\theta| \le 1$,

$r^2 \to 0$, $\ln(r^2) \to -\infty$ as $r\to 0$

$\underline{\text{i}}$ $\left| r^2\cos\theta\sin\theta\ln(r^2) \right| \le \left| r^2\ln(r^2) \right|$

$\underline{\text{ii}}$ $\lim\limits_{r\to 0} r^2\ln(r^2) = \lim\limits_{r\to 0} \dfrac{\ln(r^2)}{\frac{1}{r^2}}$ $\left(\dfrac{-\infty}{\infty} \right)$

$= \lim\limits_{r\to 0} \dfrac{\frac{2r}{r^2}}{-\frac{2}{r^3}}$ (L'Hopital's Rule)

$= \lim\limits_{r\to 0} -r^2 = 0$

By Squeeze thm, $\lim\limits_{(x,y)\to(0,0)} xy\ln(x^2+y^2) = 0$

Iterated Limit (Au 2.7)

① $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) = \lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} f(x,y) \right)$

means taking limit with $y\to 0$ first,

followed by taking limit with $x\to 0$

② Similar for $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$

$\underline{Q}$ Are they equal?

$\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) \overset{?}{=} \lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$

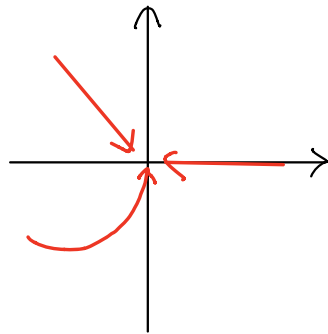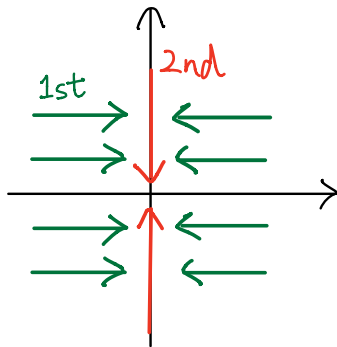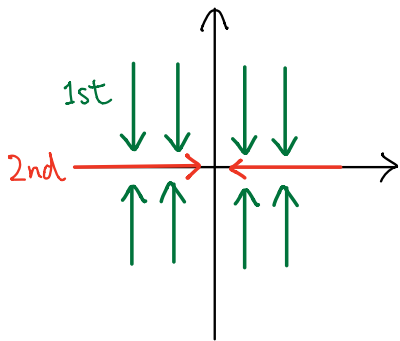$\overset{?}{=} \lim\limits_{(x,y)\to(0,0)} f(x,y)$

eg Consider $f(x,y) = \dfrac{x+y}{x-y}$.

x is fixed, like a constant

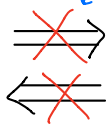$$\lim_{x\to 0}\lim_{y\to 0}\frac{x+y}{x-y} = \lim_{x\to 0}\frac{x+0}{x-0} = 1$$

Not inequal

$$\lim_{y\to 0}\lim_{x\to 0}\frac{x+y}{x-y} = \lim_{y\to 0}\frac{0+y}{0-y} = -1$$

$$\lim_{(x,y)\to(0,0)}\frac{x+y}{x-y} \quad DNE$$

1st
2nd

1st
2nd

Rmk

① $\displaystyle\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0} f(x,y)$
(Both exist and equal)

$\nRightarrow$
$\nLeftarrow$

$$\lim_{(x,y)\to(0,0)} f(x,y) \text{ exists}$$

eg. $f(x,y) = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{if } x\neq y \end{cases}$

eg. $f(x,y) = \begin{cases} x\cos\frac{1}{y} + y\cos\frac{1}{x} & \text{if } x,y\neq 0 \\ 0 & \text{if } x,y=0 \end{cases}$

② If all the three limits exist, they are equal

Continuity

Let $f: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, $\vec{a} \in A$.

Defn 1 (ε-δ) f is said to be continuous at $\vec{a}$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $x \in A$ and $\|\vec{x} - \vec{a}\| < \delta$ then $|f(\vec{x}) - f(\vec{a})| < \varepsilon$

Another equivalent definition

Defn 2 f is said to <u>continuous at $\vec{a}$</u> if $\lim\limits_{\vec{x} \to \vec{a}} f(\vec{x})$ exists and equals to $f(\vec{a})$

Defn f is said to be <u>continuous</u> if f is continuous at any point in A.

eg Let $1 \le k \le n$. Show $f(x_1, x_2, \cdots, x_n) = x_k$ is continuous on $\mathbb{R}^n$

Pf Let $\vec{a} = (a_1, a_2, \cdots, a_n)$.

For any $\varepsilon > 0$, take $\delta = \varepsilon$. If $\|\vec{x} - \vec{a}\| < \delta$, then

$$\left| f(\vec{x}) - f(\vec{a}) \right| = |x_k - a_k| \le \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2} = \|\vec{x} - \vec{a}\| < \delta = \varepsilon$$

$\therefore \lim\limits_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})$ for any $\vec{a} \in \mathbb{R}^n$

$\therefore f$ is a continuous function

Thm If $f, g: \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ are continuous at $\vec{a}$, then

① $f(x) \pm g(x)$, $k f(x)$, $f(x) g(x)$ are continuous at $\vec{a}$, where k is a constant

② $\dfrac{f(\vec{x})}{g(\vec{x})}$ is continuous at $\vec{a}$ if $g(\vec{a}) \ne 0$

Pf Follow easily from corresponding properties of limit

From the last 2 results, we have

All polynomials and rational functions $\left(\text{i.e. } \dfrac{\text{polynomial}}{\text{polynomial}}\right)$ of multi-variables are continuous

eg ① $x^3 + 3yz + z^2 - x + 7y$ is continuous on $\mathbb{R}^3$

② $\dfrac{x^3 + y^2 + yz}{x^2 + y^2}$ is continuous on

$\mathbb{R}^3 \setminus \{(x,y,z): x^2+y^2=0\} = \mathbb{R}^3 \setminus \{z\text{-axis}\}$

Let $Q(\vec{x}) = \dfrac{p_1(\vec{x})}{p_2(\vec{x})}$ be a rational function

$\vec{a}$ be a zero of $p_2(\vec{x})$ (i.e $p_2(\vec{a})=0$). Then

$Q(\vec{x})$ can be extended to a function continuous at $\vec{a}$ $\Longleftrightarrow$ $\displaystyle\lim_{\vec{x}\to\vec{a}} Q(\vec{x})$ exists

eg ① $f(x,y) = \dfrac{xy + y^3}{x^2 + y^2}$

Note $x^2 + y^2 = 0 \iff (x,y) = (0,0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{xy + y^3}{x^2 + y^2}$$

$$= \lim_{x\to 0} \frac{mx^2 + m^3 x^3}{x^2 + m^2 x^2}$$

$$= \lim_{x\to 0} \frac{m + m^3 x}{1 + m^2}$$

$$= \frac{m}{1+m^2} = \begin{cases} 0 & \text{if } m=0 \\ \frac{1}{2} & \text{if } m=1 \end{cases}$$

not equal

$\therefore$ Limit varies with slope

$\Rightarrow \displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

$\therefore$ f cannot be extended to a continuous function defined on $\mathbb{R}^2$

② $g(x,y) = \dfrac{x^4 - y^4}{x^2 + y^2}$

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}$$

$$= \lim_{r\to 0} \frac{r^4 \cos^4\theta - r^4 \sin^4\theta}{r^2 \cos^2\theta + r^2 \sin^2\theta}$$

$$= \lim_{r\to 0} r^2 \left(\cos^4\theta - \sin^4\theta\right)$$

$$= 0 \qquad \text{(Sandwich theorem)}$$

Details: $\left| r^2(\cos^4\theta - \sin^4\theta)\right| \le 2r^2$,

and $\lim\limits_{r\to 0} 2r^2 = 0$

∴ $g(x,y)$ can be extended to a continuous function on whole $\mathbb{R}^2$ with

$$g(x,y) = \begin{cases} \dfrac{x^4 - y^4}{x^2 + y^2} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Thm If $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is continuous at $\vec{a}$,
$g(x)$ is a 1-variable function continuous at $f(\vec{a})$,
then $g \circ f(\vec{x})$ is continuous at $\vec{a}$. Hence

$$\lim_{\vec{x}\to\vec{a}} g\left(f(\vec{x})\right) = g\left(\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right) = g\left(f(\vec{a})\right)$$

eg $g(x) = |x|$, $f(x_1, \cdots, x_n) = x_k$ are continuous for $1 \le k \le n$

$\Rightarrow g \circ f(x_1, \cdots, x_n) = |x_k|$ is a continuous function

eg $g(x) = |\ln x|$ is continuous on $(0, +\infty)$

$\Rightarrow g \circ f(x_1, \cdots, x_n) = |\ln x_k|$ is continuous on $\{\vec{x} \in \mathbb{R}^n : x_k > 0\}$

Rmk In particular, $\lim\limits_{(x,y)\to(0,0)} |x| = 0$, $\lim\limits_{(x,y)\to(1,0)} |\ln x| = 0$

(We used these results in examples of sandwich thm.)

eg $\sin(x^2 + yz)$, e^{x-y}, $\cos\left(\frac{1}{x^2 + y^2}\right)$, $r = \sqrt{x^2 + y^2}$
are continuous (on their domains)

Partial Derivatives $\left(\begin{array}{l}\text{Au 3.2}\\\text{Thomas 14.3}\end{array}\right)$

Study the rate of change of a function
with respect to each variable

Defn Let $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, Ω open

Then for $i = 1, 2, \cdots, n$, define the

<u>i-th partial derivative</u> of f at $\vec{x} = (x_1, \cdots, x_n) \in \Omega$

$$\frac{\partial f}{\partial x_i}(\vec{x}) = \frac{\partial f}{\partial x_i}(x_1, \cdots, x_n)$$

$$= \lim_{h \to 0} \frac{f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_i, \cdots, x_n)}{h}$$

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eg $\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$

$\frac{\partial f}{\partial y}(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$

Other
notations : $\quad \frac{\partial f}{\partial x} = \partial_1 f = D_1 f = \nabla_1 f = f_x$

$\frac{\partial f}{\partial y} = \partial_2 f = D_2 f = \nabla_2 f = f_y$

eg $f(x, y) = x^2 + y^2$

$\frac{\partial f}{\partial x} = 2x + 0 = 2x$ (Regard y as a constant)
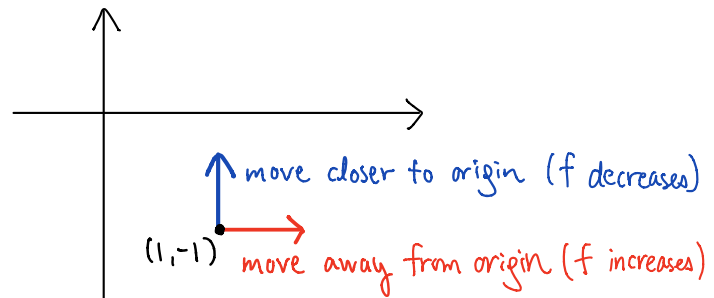
$\frac{\partial f}{\partial y} = 0 + 2y = 2y$ (Regard x as a constant)

Note $\frac{\partial f}{\partial x}(1, -1) = 2(1) = 2 > 0$

$\frac{\partial f}{\partial y}(1, -1) = 2(-1) = -2 < 0$

$\therefore$ f increases as x increases at $(1, -1)$

decreases as y increases at $(1, -1)$

Rmk Note $f(x, y) = \|(x, y)\|^2$

move closer to origin (f decreases)

$(1, -1)$ move away from origin (f increases)

eg $f(x,y,z) = xy^2 - \cos(xz)$
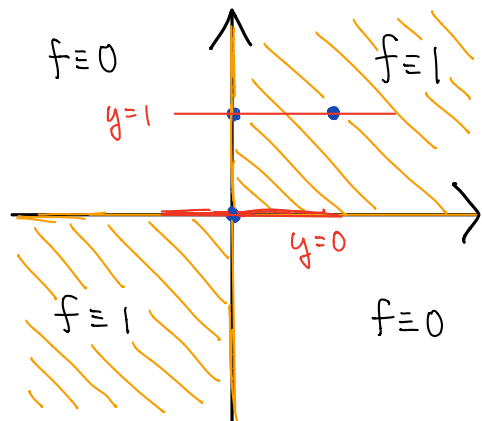
Then $f_x = y^2 + z\sin(xz)$

$f_y = 2xy + 0 = 2xy$

$f_z = 0 + x\sin(xz) = x\sin(xz)$

eg $f(x,y) = \begin{cases} 1 & \text{if } xy \geqslant 0 \\ 0 & \text{if } xy < 0 \end{cases}$

Find $\dfrac{\partial f}{\partial x}(1,1), \dfrac{\partial f}{\partial x}(0,1), \dfrac{\partial f}{\partial x}(0,0)$
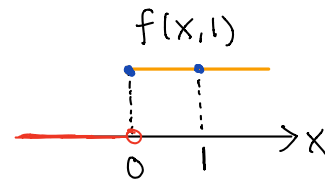
Sol

f ≡ 0   f ≡ 1
y=1
y=0
f ≡ 1   f ≡ 0

$\dfrac{\partial f}{\partial x}$: Fix y, differentiate $f(x,y)$ w.r.t. x

Along $y=1$

$f(x,1) = \begin{cases} 1 & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$

$f(x,1)$

$\Rightarrow \dfrac{\partial f}{\partial x}(1,1) = 0, \quad \dfrac{\partial f}{\partial x}(0,1)$ DNE

Along $y=0$

$f(x,0) \equiv 1, \quad \forall x \in \mathbb{R}$

$\Rightarrow \dfrac{\partial f}{\partial x}(0,0) = 0$

Rmk Similarly $\dfrac{\partial f}{\partial y}(0,0) = 0$

Note f is not continuous at $(0,0)$

$\therefore \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ exist at $(0,0) \not\Rightarrow f$ is continuous at $(0,0)$

Higher order partial derivative

eg Consider $f(x,y)$

1st order derivative : $\dfrac{\partial f}{\partial x} = f_x$, $\dfrac{\partial f}{\partial y} = f_y$

2nd order derivatives : $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$

Differentiate w.r.t. x first $\rightarrow$ $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}$ $\quad$ x first $\downarrow$

y first $\rightarrow$ $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}$ and $\dfrac{\partial^2 f}{\partial y^2} = f_{yy}$

y first

3rd order derivatives : $\dfrac{\partial^3 f}{\partial x \partial y^2} = \dfrac{\partial}{\partial x}\left[\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)\right] = \left[(f_y)_y\right]_x = f_{yyx}$

$\dfrac{\partial^3 f}{\partial x^3} = f_{xxx}$

Others : f_{xyy} , f_{yxy} , f_{yyy} , f_{xxy} , f_{xyx} , f_{yxx}

Rmk Similar for higher order / more variables

eg f_{xxyx} , g_{xyz} for $g(x,y,z)$

eg $f(x,y) = x\sin y + y^2 e^{2x}$

Find all 1st, 2nd order partial derivatives

Sol

$f_x = \sin y + 2y^2 e^{2x}$

$f_y = x\cos y + 2y e^{2x}$

$f_{xx} = (f_x)_x = 4y^2 e^{2x}$

$f_{xy} = (f_x)_y = \cos y + 4y e^{2x}$

$f_{yx} = (f_y)_x = \cos y + 4y e^{2x}$ $\quad$ equal. Coincidence?

$f_{yy} = (f_y)_y = -x\sin y + 2 e^{2x}$

<u>Q</u> Is it always true that $f_{xy} = f_{yx}$?

<u>A</u> No, a counter example below:

<u>eg</u> Compute $f_{xy}(0,0)$, $f_{yx}(0,0)$, where

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$
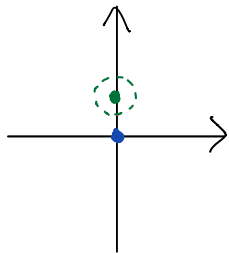
<u>Sol</u> $f_{xy} = (f_x)_y$

$\Rightarrow f_{xy}(0,0) = \lim\limits_{k \to 0} \dfrac{f_x(0,k) - f_x(0,0)}{k}$

Need to find $f_x(0,k)$ for $k \neq 0$ and $f_x(0,0)$

For $k \neq 0$,

$f = \dfrac{xy(x^2-y^2)}{x^2+y^2}$ near $(0,k)$

$f_x = \dfrac{(x^2+y^2)(3x^2y - y^3) - xy(x^2-y^2)(2x)}{(x^2+y^2)^2}$ near $(0,k)$

$\Rightarrow f_x(0,k) = \dfrac{k^2(-k^3) - 0}{k^4} = -k$

$f_x(0,0) = \lim\limits_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h}$

$\quad\quad = \lim\limits_{h \to 0} \dfrac{0-0}{h} = 0$

$f_{xy}(0,0) = \lim\limits_{k \to 0} \dfrac{f_x(0,k) - f_x(0,0)}{k}$

$\quad\quad = \lim\limits_{k \to 0} \dfrac{-k-0}{k} = -1$

Similar calculation gives $f_{yx}(0,0) = 1$

$\left(\begin{array}{l} \text{Alternatively, note } f(x,y) = -f(y,x) \\ \Rightarrow f_{yx}(0,0) = -f_{xy}(0,0) = -1 \end{array} \right)$

In this example, $f_{xy}(0,0) \neq f_{yx}(0,0)$

<u>Q</u> When do $f_{xy} = f_{yx}$?

<u>Thm</u> (Clairaut's Theorem / Mixed Derivative Thm)

Let $\Omega \subseteq \mathbb{R}^2$ be an open set, $f: \Omega \to \mathbb{R}$

If f_{xy}, f_{yx} exist and are continuous on Ω,

then $f_{xy} = f_{yx}$ on Ω
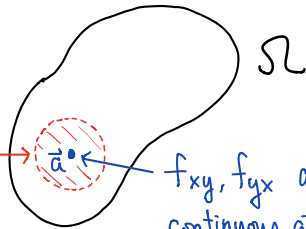

<u>Rmk</u> We will prove a stronger version:

Let $\Omega \subseteq \mathbb{R}^2$ be open, $f: \Omega \to \mathbb{R}$, $\vec{a} \in \Omega$. Suppose

① f_{xy}, f_{yx} exist in an open disc containing $\vec{a}$

② f_{xy}, f_{yx} are continuous at $\vec{a}$.

Then $f_{xy}(\vec{a}) = f_{yx}(\vec{a})$

f_{xy}, f_{yx} exist on $\longrightarrow$ Ω
$B_\varepsilon(\vec{a})$ for some $\varepsilon > 0$ — f_{xy}, f_{yx} are continuous at $\vec{a}$
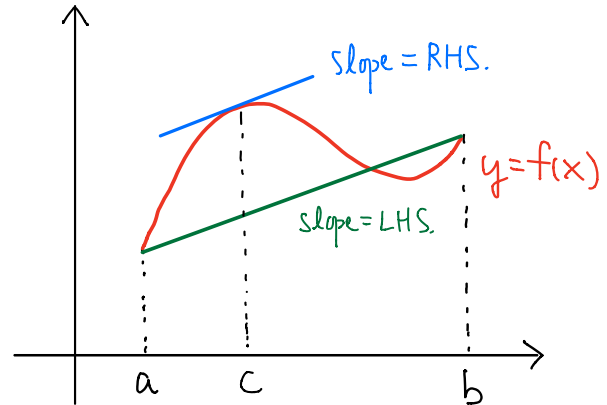

Tool for the proof:

<u>Mean Value Theorem for 1-variable function</u>

Let $f: [a,b] \to \mathbb{R}$, continuous on $[a,b]$
   differentiable on (a,b)

Then $\exists c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$
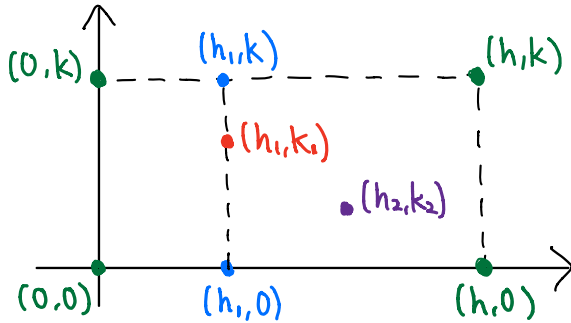

Slope = RHS.

$y = f(x)$

Slope = LHS.

Pf of Clairaut's Theorem

We may assume $\vec{a} = (0,0) \in \Omega$

Need to show $f_{xy}(0,0) = f_{yx}(0,0)$

Let $h, k > 0$ and $[0,h] \times [0,k] \subseteq \Omega$

$\alpha = f(h,k) - f(0,k) - f(h,0) + f(0,0)$

Let $g(x) = f(x,k) - f(x,0)$, $0 \le x \le h$

Then $\alpha = g(h) - g(0)$

$g'(x) = f_x(x,k) - f_x(x,0)$

MVT $\Rightarrow \exists h_1 \in (0,h)$ such that

$$\frac{\alpha}{h} = \frac{g(h) - g(0)}{h} = g'(h_1) = f_x(h_1,k) - f_x(h_1,0)$$

MVT again $\Rightarrow \exists k_1 \in (0,k)$ such that

$$\frac{f_x(h_1,k) - f_x(h_1,0)}{k} = f_{xy}(h_1,k_1) \qquad \textit{like a MVT for } f(x,y)$$

$\therefore \alpha = h\left[f_x(h_1,k) - f_x(h_1,0) \right] = hk\, f_{xy}(h_1,k_1)$

Similarly, $\exists (h_2, k_2) \in (0,h) \times (0,k)$ such that

$$\alpha = hk\, f_{yx}(h_2,k_2)$$

$\therefore f_{xy}(h_1,k_1) = f_{yx}(h_2,k_2)$ ⊛

Take $h, k \to 0^+$, then $(h_1,k_1), (h_2,k_2) \to (0,0)$

⊛ and continuity of f_{xy}, f_{yx} at $(0,0)$

$\Rightarrow f_{xy}(0,0) = f_{yx}(0,0)$